THE UNIVERSITY OF ALBERTA

BITOPOLOGICAL FUNCTION SPACES

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Bitopological Function Spaces," submitted by YONG-WOON KIM in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Kelly introduced the concept of a bitopological space, i.e., a triple (X, L_1, L_2) where X is a set and L_1, L_2 are topologies on X. He defined pairwise Hausdorff, pairwise regular, pairwise normal spaces and obtained generalizations of several standard results such as Urysohn's lemma, Tietze's extension theorem, Urysohn's metrization theorem and the Baire category theorem. Fletcher and Lane independently defined pairwise completely regular and pairwise uniform spaces and proved their equivalence.

This thesis began in an attempt to define the concept of pairwise compactness in a bitopological space, in a non-trivial way. After recalling known definitions and results in Chapter 1, this is done in Chapter 2. It is shown that the definition used here satisfies most of the requirements. Furthermore, maximal and minimal bitopological spaces are investigated and the results are used in the sequel. The results are then applied in Chapter 3 to function space topologies which are studied in detail. These function spaces are studied not only for pairwise continuous functions but also for certain types of non-continuous functions such as pairwise connected and pairwise almost continuous functions.

Analogously in bitopological space, connected open topology and graph topology are considered in Chapter 4, 5, 6.

Finally in the last chapter a new function space is introduced which is especially useful for the space of all functions which have at worst discontinuity of the first kind. This sheds more light on the Skorokhod M-convergence.

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TABLE OF CONTENTS

		Page
	ABSTRACT	(i)
	ACKNOWLEDGEMENTS	(iii)
CHAPTER 1:	PRELIMINARY	1
CHAPTER 2:	p-COMPACTNESS	6
CHAPTER 3:	p-COMPACT OPEN TOPOLOGY	18
CHAPTER 4:	(12)-CONNECTED OPEN TOPOLOGY	30
CHAPTER 5:	p-CONNECTED OPEN TOPOLOGY	37
CHAPTER 6:	BIGRAPH TOPOLOGY	41
CHAPTER 7:	ALMOST CONVERGENT TOPOLOGY	48
	NOTATIONS AND CONVENTIONS	56
	BIBLIOGRAPHY	60

CHAPTER 1

PRELIMINARY

In this chapter we discuss elementary notations, definitions and some of the known results concerning bitopological spaces that are used in the following chapters.

(1.1) <u>Definition:</u> (Kelly [11] A space X on which are defined two topologies L_1 and L_2 is called a <u>bitopological space</u>, and is denoted by (X, L_1, L_2).

The following definition was given by Weston [26], who used the term consistent.

(1.2) <u>Definition:</u> (Weston [26]) A bitopological space (X, L_1, L_2) is <u>pairwise Hausdorff</u> iff $x, y \in X$ and $x \neq y$ implies there exist $U \in L_1$, $V \in L_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

We will use the symbol p- to denote any pairwise property: e.g. p-Hausdorff stands for pairwise Hausdorff.

(1.3) <u>Definition:</u> (Weston [26]) In a bitopological space (X, L_1, L_2), we say that L_1 <u>is coupled</u> to L_2 iff for all $G \in L_1$, $\bar{G} \subset \bar{\bar{G}}$, where $\bar{G}$, $\bar{\bar{G}}$ denote the closures of G in L_1, L_2, respectively.

(1.4) Lemma: (Weston [26]) If (X, L_1, L_2) is p-Hausdorff and L_1 is coupled to L_2, then L_1 is Hausdorff.

(1.5) Definition: (Kelly [11]) A bitopological space (X, L_1, L_2) is (12)-regular iff for each $x \in X$ there exists an L_1-neighbourhood base of L_2-closed sets. If, in addition, it is (21)-regular then it is p-regular.

Kelly [11] defined p-normal bitopological spaces in an analogous manner, namely

(1.6) Definition; (Kelly [11]) A bitopological space (X, L_1, L_2) is said to be p-normal iff for any L_i-closed set A and L_j-closed set B with $A \cap B = \phi$ there exist L_j-open set U and L_i-open set V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$ $(i \neq j,\ i = 1,2)$.

One motivation for the study of bitopological spaces is given by considering a "pseudo-quasi metric" (or (p-q) metric).

(1.7) Definition: A p-q metric on a set X is a non-negative real-valued function p on the product $X \times X$ such that

(i) $p(x,x) = 0$

(ii) $p(x,z) \leq p(x,y) + p(y,z)$ $(x,y,z, \in X)$ and if q is defined as

$$q(x,y) = p(y,x)$$

then q is said to be the conjugate p-q metric of p. If further p

satisfies the condition,

(iii) $p(x,y) = 0$ only if $x = y$, $(x,y \in X)$ then p is said to be a <u>quasi metric</u>.

The following is a generalization of Urysohn's metrization theorem.

(1.8) <u>Theorem:</u> (Kelly [11]) If (X, L_1, L_2) is p-regular satisfying C_2, then (X, L_1, L_2) is p-q metrizable. If, in addition, it is p-Hausdorff, then it is quasi metrizable.

(1.9) <u>Theorem:</u> (kelly [11]) If L_1 and L_2 are topologies on X determined by p-q metric p and its conjugate q, then (X, L_1, L_2) is p-regular and p-normal and (X, L_1, L_2) is p-Hausdorff iff p and q are quasi metric.

(1.10) <u>Definition:</u> If f is a function from X into R^*, then f is LSC (resp. USC) whenever $\{ x \in X : f(x) \leq a \}$ (resp. $\{ x \in X : f(x) \geq a \}$ is closed for each $a \in R^*$, where R^* is the extended real numbers with the usual topology.

The notation "LSC" and "USC" are used for lower and upper semi-continuous, respectively. Further, "ULX" denotes the set of all real-valued functions on X that are both L_1-USC and L_2-LSC. "LUX" is defined analogously.

The following is a generalization of Urysohn's Lemma.

(1.11) <u>Theorem:</u> (Kelly [11]) If (X, L_1, L_2) is p-normal, then for

any L_2-closed set F and L_1-closed set H with $F \cap H = \phi$ there exists a $g \in ULX$ such that

$$g(x) = 0 \quad \text{on } F$$

$$g(x) = 1 \quad \text{on } H$$

$$0 \leq g \leq 1 \quad \text{on } X .$$

Császár [3] was first to prove that every topological space is quasi-uniformizable. Subsequently, Pervin [18] gave a direct proof of this result. Reference [15] contains a systematic exposition of quasi uniform spaces. Fletcher [6] and Lane [13] independently defined p-completely regular spaces and generalized the classical form of the uniformization theorem.

(1.12) Definition: (Fletcher [6]) (X, L_1, L_2) is (12)-completely regular iff for any L_1-closed set F and $x \notin F$ there exists $f \in ULX$ such that

$$f(x) = 1$$

$$f(y) = 0 \quad \text{for } y \in F \text{ and } 0 \leq f \leq 1 \text{ on } X$$

and it is p-completely regular iff (12) and (21)-completely regular.

Since, if L_i is T_1 for $i = 1,2$, then a singleton is L_i-closed, therefore we have

(1.13) Theorem: If (X, L_1, L_2) is p-normal and L_i is T_1 for $i = 1,2$, then (X, L_1, L_2) is p-completely regular.

(1.14) Corollary: If (X, L_1, L_2) is p-normal and p-Hausdorff, then

(X, L_1, L_2) is p-completely regular.

(1.15) Definition: (Fletcher [6]) (X, L_1, L_2) is p-uniform iff there exists a quasi-uniformity $\mathcal{U}$ such that $L_1 = T(\mathcal{U})$ and $L_2 = T(\mathcal{U}^{-1})$, where $T(\mathcal{U})$ denotes the topology on X induced by $\mathcal{U}$ and $\mathcal{U}^{-1} = \{ U^{-1} : U \in \mathcal{U} \}$.

(1.16) Theorem: (Fletcher [6]) (X, L_1, L_2) is p-uniform iff it is p-completely regular.

The above result is a generalization of the classical result of A. Weil, namely that a topological space is uniformizable iff it is completely regular.

CHAPTER 2

p-COMPACTNESS

In a private communication Fletcher suggested the problem of suitably defining p-compactness for bitopological spaces. Such a definition should meet the following requirement (i) a bitopological space (X, L_1, L_2) can be p-compact without $L_1 = L_2$; (ii) p-compactness and p-Hausdorff together imply p-normality and (iii) p-compactness is a p-continuous invariant.

We shall find later that such a definition of p-compactness also helps in the study of bitopological function spaces. In this chapter we state the definition of p-compactness and then develop some results which will be used in the sequel.

(2.1) Definition: Let (X, L_1, L_2) be a bitopological space. For any $V \in L_2$ where $V \neq \emptyset$, consider $\{U_i \cup V\}$ where $U_i \in L_1$. Then $L_1(V) \equiv \{X, \emptyset, \{U_i \cup V\}\}$ is a topology and is said to be an adjoint topology of L_1 with respect to V .

(2.2) Definition: (X, L_1, L_2) is said to be (12)-compact iff for each non-empty $V \in L_2$, $L_1(V)$ is compact. This is equivalent to V^c being L_1-compact for each non-empty $V \in L_2$. If, in addition, it is (21)-compact we say that (X, L_1, L_2) is p-compact.

<u>Remark:</u> In a (12)-compact space $L_1 \subset L_2$ implies (X, L_1) is compact.

(2.3) <u>Example:</u> Let X be the set of all real numbers, L_1 be the topology generated by the sets $\{(-\infty, a)\}$ and L_2 be generated by the sets $\{(a, \infty)\}$, where a is any real number. Then (X, L_1, L_2) is p-compact but neither L_1 nor L_2 is compact.

(2.4) <u>Example:</u> Let $X = \{N \cup Ni\}$, where N is the set of all natural numbers and $i^2 = -1$. Let L_1 be generated by $\{N \cup Fi\}$ where F is a finite subset of N and $L_2 = \{\emptyset, X, N, Ni\}$. Here (X, L_2) is compact but (X, L_1, L_2) is not (12)-compact.

(2.5) <u>Example:</u> Let X be the set in Example (2.4). Let L_1 be generated by $(N - F) \cup G$, where F is a finite subset of N and G an arbitrary subset of Ni. Let L_2 be generated by $H \cup (Ni - Fi)$ where F is as before and H an arbitrary subset of N. Here (X, L_1, L_2) is p-Hausdorff and p-compact, but (X, L_i) is not compact for $i = 1,2$.

(2.6) <u>Theorem:</u> If (X, L_1, L_2) is L_i-compact for $i = 1,2$ and p-Hausdorff, then (X, L_1, L_2) is (ij)-compact, $i,j = 1,2$.

<u>Proof:</u> If $V \in L_j$. Then V^c is L_j-compact and since X is p-Hausdorff it is L_i-closed. Since (X, L_i) is compact, V^c is L_i-compact, showing that (X, L_1, L_2) is (ij)-compact for $i,j = 1,2$.

Remark: We can easily deduce the following statements from (2.6):

(1) If (X, L_1, L_2) is p-Hausdorff and L_i , $i = 1,2$, is compact then $L_1 \equiv L_2$.

(2) If (X, L_1, L_2) is p-Hausdorff and L_i , $i = 1,2$, is compact then (X, L_1, L_2) is p-compact.

The following example shows that p-T_2 is not superfluous in the hypothesis.

Let $X = [0,1]$. $L_1 = \{\{0\}, \{[0, a) : a \in [0, 1]\}, X, \emptyset\}$ while L_2 is generated by $\{ \{1\}, \{(a, 1] : a \in [0, 1]\}, X , \emptyset\}$, since for $1/2, 2/3 \in X$, we cannot have $U \in L_1$, $V \in L_2$ such that $2/3 \in U$, $1/2 \in V$ and

$$U \cap V = \emptyset$$

(X, L_1, L_2) is not p-Hausdorff. However, $(X, L_1 L_2)$ is compact with respect to L_i, $i = 1,2$, because every open covering of X with L_i-open set must include X. Moreover, this is not p-compact for $\{1\} \in L_2$ and $\{1\}^c = [0, 1)$ is not an L_1-compact subset.

(2.7) Definition: A mapping f

$$f : (X, L_1, L_2) \to (Y, S_1, S_2)$$

is said to be p-continuous iff $f : (X, L_i) \to (Y, S_i)$ is continuous for $i = 1,2$.

(2.8) Lemma: (12)-compactness is a p-continuous invariant.

Proof: Consider $f : (X, L_1, L_2) \to (Y, S_1, S_2)$, where f is p-continuous

and (X, L_1, L_2) is (1,2)-compact. Let

$$f(X) \subset \{\cup U_i \cup V\}, \quad U_i \in S_1 \text{ and } V \in S_2, \quad V \cap f(X) \neq \emptyset .$$

then

$$X \subset \cup f^{-1}(U_i) \cup f^{-1}(V)$$

Since f is p-continuous, $f^{-1}(U_i) \in L_1$ and $f^{-1}(V) \in L_2$. By (12)-compactness of X there exists a finite subclass $\{U_{ij} : n \geq j \geq 1\}$

$$X \subset \bigcup_{j=1}^{n} f^{-1}(U_{ij}) \cup f^{-1}(V)$$

which implies

$$f(X) \subset \bigcup_{j=1}^{n} U_{ij} \cup V .$$

(2.9) Corollary: p-compactness is a p-continuous invariant.

(2.10) Lemma: Let (X, L_1, L_2) be a (12)-compact space.

(i) If C is a L_2-closed proper subset, then C is L_1-compact and (12)-compact.

(ii) If C is L_1-closed, then C is (12)-compact.

Proof: Consider $C \subset \cup U_i$, $U_i \in L_1$, where C is L_2-closed. Then $X \subset \cup U_i \cup C^c$, $C^c \neq \emptyset$. But $C^c \in L_2$. Therefore, there exists finite subcollection $\{U_{ij} : j = 1, \ldots n\}$

$$X \subset \bigcup_{j=1}^{n} (U_{ij} \cup C^c) .$$

Now $C = X - C^c \subset \bigcup_{j=1}^{n} U_{ij}$ which implies C is L_1-compact. To show C is (12)-compact, consider $C \subset \bigcup U_i \cup V$, where $U_i \in L_1$, $V \in L_2$ and $V \cap C \neq \emptyset$. Then

$$X = \bigcup U_i \cup V \cup C^c .$$

But

$$V \cup C^c \in L_2 .$$

Let $V' = V \cup C^c$. Since X is (12)-compact, there exists finite subclass $\{ U_{ij} : j = 1,\dots n\}$ such that

$$X = \bigcup_{j=1}^{n} U_{ij} \cup V'$$

Now $C = X - C^c \subset \bigcup_{j=1}^{n} U_{ij} \cup V$ which implies C is (12)-compact. Part (ii) is proved analogously.

(2.11) <u>Corollary:</u> If (X, L_1, L_2) is p-compact, then an L_i-closed subset C is p-compact and if C is a proper subset of X then C is L_j-compact $(i \neq j,\ i = 1,2)$.

(2.12) <u>Theorem:</u> Let (X, L_1, L_2) be p-Hausdorff.

(i) If L_2 is Hausdorff, then a (12)-compact subset of X is L_2-closed.

(ii) If L_1 is Hausdorff, then a (12)-compact subset of X is L_1-closed.

Proof: Assume L_2 is Hausdorff and C is a (12)-compact subset of X with $x \notin C$. Then, by the virture of the p-Hausdorff property, for each $y_i \in C$ there exist $V_{y_i} \in L_2$ and $U_{y_i} \in L_1$ with $x \in V_{y_i}$ and $y_i \in U_{y_i}$ such that $V_{y_i} \cap U_{y_i} = \emptyset$. We also have for every $y \in C$ $V_y \in L_2$ and $U_y \in L_2$ such that $V_y \cap U_y = \emptyset$ where $x \in V_y$ and $y \in U_y$. Since C is a (12)-compact subset there exists $n \in N$ such that

$$C \subset \bigcup_1^n (V_{y_i} \cup U_y) .$$

Let $V = (\bigcap_1^n V_{y_i}) \cap V_y$. Then $V \in L_2$ and $x \in V$ with $V \cap C = \emptyset$.

Similarly, we can show (ii)

The following is proved analogously.

(2.13) Theorem: If (X, L_1, L_2) is p- Hausdorff and $K \subset X$ is L_i-compact, then K is L_j-closed $(i \neq j \ , \ i = 1,2)$.

Remark: If (X, L_1, L_2) is p-compact and L_1 and L_2 are Hausdorff then $L_1 = L_2$ because if $U \in L_i$ then U^c is L_j-compact. By Hausdorff property U^c is L_j-closed which implies $U \in L_j$.

(2.14) Corollary: Let (X, L_1, L_2) be p-Hausdorff.

(i) If L_i is coupled to L_j then every (ji)-compact subset of X is L_i-closed.

Proof: Use Lemma (1.4) and Theorem (2.12).

(2.15) Definition: A subset $C \subset X$ is said to be (12)*-separated iff for each $x \in C$ and $y \in C^c$ there exist $U_x \in L_1$ and $V_y \in L_2$ such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \emptyset$. A set both (12)*-separated and (21) separated is called P*-separated.

In Example (2.3) every L_2-closed set is (12)*-separated and every L_1-closed set is (21)*-separated. If (X, L_1, L_2) is p-Hausdorff, then every subset of X is p*-separated.

The following is easily shown.

(2.16) Lemma: If an L_j-compact subset K of $(X, L_1 L_2)$ is (ji)*-separated, then K is L_i-closed $(i \neq j\ ,\ i = 1,2)$.

Remark: (2.13) is easily deduced from the above lemma.

(2.17) Theorem: If $(X, L_1 L_2)$ is (12)-compact and if every L_2-closed set is (12)*-separated, then (X, L_1, L_2) is (21)-regular.

Proof: Let C be an L_2-closed subset of X . Then C is L_1-compact by Lemma (2.10). Assume $p \notin C$. By (12)*-separateness for each $y_i \in C$ there exists $V_{y_i} \in L_2$ and $U_{y_i} \in L_1$ such that $y_i \in U_{y_i}$ and $p \in V_{y_i}$ with $U_{y_i} \cap V_{y_i} = \emptyset$. Since C is L_1-compact we have $n \in N$ such that $C \subset \bigcup_1^n U_{y_i}$. Let $V_p = \bigcap_1^n V_{y_i}$. Then

$$V_p \cap \left(\bigcup_1^n U_{y_i} \right) = \emptyset , \qquad U = \bigcup_1^n U_{y_i} \in L_2$$

where $U \supset C$ and $p \in V_p$.

(2.18) Corollary: If (X, L_1, L_2) is p- compact and if every L_i-closed set is $(ji)^*$-separated, then (X, L_1, L_2) is p-regular, $i \neq j$, $i = 1,2$.

(2.19) Corollary: If (X, L_1, L_2) is p-compact and p-Hausdorff, then (X, L_1, L_2) is p-regular.

(2.20) Theorem: If (X, L_1, L_2) is p-compact and if every L_i-closed set is $(ji)^*$-separated, then (X, L_1, L_2) is p-normal $(i \neq j\ ,\ i =1,2)$.

Proof: Assume $C \subset X$ is L_1-closed (we have an equivalent result if C is L_2-closed). Let $C \subset A$ where A is L_2-open. We have to show there exist L_2-open set U and L_1-closed B such that

$$C \subset U \subset B \subset A$$

which is an equivalent form of (1.6). Since C is L_1-closed by (2.10) C is L_2-compact. For each $x_i \in C$ there exist U_{y_i}, an L_2-open neighborhood of x_i, and an L_1-closed B_i such that

$$U_{x_i} \subset B_i \subset A$$

by (2.18). Since C is L_2-compact there exist a finite number of sets $\{U_{x_{ij}}\}_1^n$ such that

$$C \subset \bigcup_{j=1}^{n} U_{x_{ij}} = U .$$

Let $B = \bigcup_{1}^{n} B_{x_{ij}}$. Therfore

$$C \subset U \subset B \subset A$$

(2.21) <u>Corollary:</u> If (X, L_1, L_2) is p-compact and p-regular, then (X, L_1, L_2) is p- normal.

(2.22) <u>Corollary:</u> If (X, L_1, L_2) is p-compact and p-Hausdorff, then (X, L_1, L_2) is p-normal.

<u>Proof:</u> By (2.19) p-compactness and p-Hausdorff together imply p-regularity, so that the result follows from (2.21) .

(2.23) <u>Theorem:</u> If (X, L_1, L_2) is p-compact and p-Hausdorff, then (X, L_1, L_2) is p-completely regular.

<u>Proof:</u> Apply (2.22) and (1.14).

(2.24) <u>Theorem:</u> If (X, L_1, L_2) is p-compact and p-Hausdorff, then (X, L_1, L_2) is p-uniform .

<u>Proof:</u> Apply (2.23) and (1.16).

(2.25) <u>Definition:</u> A partial order $\geq$ on the family of all bitopological spaces is defined as follows:

$$(X, L_1, L_2) \geq (X, S_1, S_2) \text{ iff } L_1 \supset S_1 \text{ and } L_2 \supset S_2 .$$

(2.26) Lemma: If (X, L_1, L_2) is (12)-compact and p-Hausdorff, then for $L_2' \subsetneq L_2 \subsetneq L_2''$:

(1) (X, L_1, L_2') is not p-Hausdorff, and

(2) (X, L_1, L_2'') is not (12)-compact

Proof: Assume (X, L_1, L_2') is p-Hausdorff. Let $G \in L_2$. Then G^c is L_2-closed and hence L_1-compact by(2.10). Since an L_1-compact subset in a p-Hausdorff bitopological space (X, L_1, L_2') is L_2'-closed, $G \in L_2'$. Therefore, $L_2 = L_2'$, which is a contradiction.

Now assume that (X, L_1, L_2'') is (12)-compact and $G \in L_2''$. Then G^c is L_2''-closed in the (12)-compact space, which implies G^c is L_1-compact. Since (X, L_1, L_2) is p-Hausdorff, G^c is L_2-closed and $G \in L_2$; this implies that $L_2 = L_2''$, again a contradiction.

(2.27) Theorem: If (X, L_1, L_2) is p-Hausdorff and p-compact, then for $L_i' \subsetneq L_i \subsetneq L''_i$

(1) (X, L_i', L_j) is not p-Hausdorff and

(2) (X, L_i'', L_j) is not p-compact, $i,j = 1,2$ and $i \neq j$

(2.28) Definition: (X, L_1, L_2) is said to be p- $T_{1\frac{1}{2}}$ or p-semi Hausdorff iff for all $x,y \in X$, $x \neq y$, there exist $U_x \in L_i$ and

$V_y \in L_j$, $i \neq j$ with i either 1 or 2 , such that $x \in U_x$, $y \in V_y$ and

$$U_x \cap V_y = \emptyset$$

Remark: p- T_2 implies p-$T_{1\frac{1}{2}}$ but p- $T_{1\frac{1}{2}}$ does not imply p- T_1 , however it is p- T_0. The space of Example (2.3) is p- $T_{1\frac{1}{2}}$ but not p- T_2 .

The following example shows that Theorem (2.27) is not true if p-Hausdorff is replaced by p- $T_{1\frac{1}{2}}$.

(2.29) Example: Let X be the interval $[0, 1]$ with the following topology: L_1 is generated by $\{(b, 1]\}$ while L_2 is generated by $\{[0, a)\}$ where $0 \leq a$, $b \leq 1$. Let $L_1' = L_2'$ be generated by $\{(a, b)\}$. Then both bitopological space (X, L_1, L_2), (X, L_1', L_2') are p-compact and p- $T_{1\frac{1}{2}}$ and yet $L_i \subsetneq L_i'$.

(2.30) Theorem: If $f : (X, L_1, L_2) \to (Y, S_1, S_2)$ is one-to-one onto, p-continuous, Y is p-Hausdorff and X is p-compact, then f is a p-homeomorphism, (i.e. a one-to-one onto p-continuous p-open map).

Proof: Consider (X, L_1', L_2') where $L_i' = \{f^{-1}(U) : U \in S_i\}$ for $i = 1,2$. Then $L_i' \subset L_i$ and (X, L_1', L_2') is p-compact and p-Hausdorff, and so is (X, L_1, L_2) . By Theorem (2. 27) we have $L_i' = L_i$, $i = 1,2$ which implies that f is a p-homeomorphism.

In Theorem (2.30) if we assume that $L_1 = L_2$ and $S_1 = S_2$ then we have the well-known theorem:

(2.31) <u>Corollary</u>: If $f : (X, L) \to (Y, S)$ is onto, one-to-one, continuous, Y is Hausdorff and X is compact, then f is a homeomorphism.

CHAPTER 3

p-COMPACT OPEN TOPOLOGY

In this chapter, using the previous results, we discuss the p-compact open topology which is a generalization of the compact open topology. We prove analogously in bitopological spaces of the results concerning compact open or K topology.

To have meaningful results, we assume that (X, L_1, L_2) is a p-Hausdorff space. First we define the p-compact open topology, or p-K topology, and then study this topology on $C_{S_{12}}^{L_{12}}$, where $C_{S_{12}}^{L_{12}}$ is the set of all p-continuous functions $f : (X, L_1, L_2) \to (Y, S_1, S_2)$.

(3.1) <u>Definition:</u> For each pair of sets $A \subset X$, $B \subset Y$, let

$$\widetilde{K}(A,B) = \{f \in Y^X : f(A) \subset B\} .$$

A <u>p-K topology</u> on Y^X is the pair of topologies ψ_1 and ψ_2 which are generated by all sets of the form $\{\widetilde{K}(A,V) : V \in S_1\}$ and $\{\widetilde{K}(A,U) : U \in S_2\}$ (where A is p- compact) respectively.

Then

$$(Y^X : \psi_1, \psi_2)$$

is said to be <u>a p-K topological space</u>. Similarly we can have p-pointwise

convergent topology consisting of a pair of topology P_i, $i = 1,2$.

The following lemmas are frequently used in the sequel.

(3.2) Lemma: If (X, L_1, L_2) is p-Hausdorff, then every non-empty p-compact subset $K \subset X$ can be expressed as

$$K = K_1 \cup K_2$$

where K_1 is a non-empty L_1-compact and K_2 is a non-empty L_2-compact subset of X.

Proof: Let K be a p-compact subset of X.

Case (i): If K is a singleton set then $K_1 = K$ and $K_2 = K$ are both L_1- and L_2- compact subsets and the statement is true:

Case (ii): Suppose there exist $x,y \in K$, $x \neq y$. Since (X, L_1, L_2) is p-Hausdorff, there exist $U \in L_1$ and $V \in L_2$ such that $x \in U$ and $y \in V$ with $U \cap V = \emptyset$. Since K is p-compact,

$K_1 = K - V$ is non-empty L_1-compact,

$K_2 = K - U$ is non-empty L_2-compact

and

$$K_1 \cup K_2 = (K - V) \cup (K - U) = K - (U \cap V) = K .$$

Similarly we can show:

(3.3) <u>Lemma:</u> In a bitopological space (X, L_1, L_2) if L_1, L_2 are both Hausdorff, then every non-empty p-compact subset $K \subset X$ can be expressed as

$$K = K_1 \cup K_2$$

and

$$K = K_1' \cup K_2'$$

where K_1, K_2 are non-empty L_1-compact and K_1', K_2' are non-empty L_2-compact subsets of X .

<u>Remark:</u> The above lemma implies that a p-compact subset K is compact subset with respect to L_i whenever each L_i is Hausdorff for L_i , $i = 1,2$.

(3.4) <u>Lemma:</u> In a bitopological space (X, L_1, L_2) , if C is L_i-closed, $i = 1$ or 2 , and $K \subset X$ is p-compact, then

$$T = C \cap K$$

is a p-compact subset.

<u>Proof:</u> If $T = C \cap K = \emptyset$, then the result is obvious and therefore we assume $T \neq \emptyset$. let $\{U_i \cup V\}$ be an $L_1(V)$-covering of T , where $V \in L_2$ and $L_1(V)$ is the adjoint topology of L_1 with respect to V . $T \subset \bigcup_{i \in A} (U_i \cup V)$, where A is an index set and $T \cap V \neq \emptyset$.

Case (i): If C is L_1-closed, then $\{(C^c \cup U_i) \cup V\}$ is an

$L_1(V)$-covering of K with $V \cap K \neq \emptyset$. Since K is a p-compact subset of (X, L_1, L_2) there exists a finite subcover

$$K \subset \bigcup_{j=1}^{n} (C^c \cup U_{ij}) \cup V$$

which implies

$$T = K \cap C \subset \bigcup_{1}^{n} [(C^c \cup U_i) \cup V] \cap C \subset \bigcup_{1}^{n} (U_i \cup V)$$

Therefore T is a p-compact subset of X .

Case (ii) is similarly handled.

From now on we consider only the set $C^{L_{12}}_{S_{12}}$. Concerning the relation between the p-K topology and the K-topology we have the following:

(3.5) <u>Theorem</u>: If (X, L_1, L_2) is a bitopological space that both L_1 and L_2 are Hausdorff, then

$$\psi_i \subset \kappa_i \quad , \quad i = 1,2$$

where κ_i is the compact open topology on the set of all continuous functions on (X, L_i) to (Y, S_i) .

<u>Proof</u>: Let K be a p-compact subset of X . Then by (3.3) $K = K_1 \cup K_2$, where K_i is L_1-compact, $i = 1,2$, and $K = K_1' \cup K_2'$, with L_2-compact K_i', $i = 1,2$. For any $\widetilde{K}(K,V) \in \psi_i$

$$K(K,V) = \widetilde{K}(K_1,V) \cap \widetilde{K}(K_2,V)$$

where $V \in S_i$, which implies

$$\widetilde{K}(K,V) \in \kappa_i(K\text{-topology}) .$$

We have a non-trivial example which show the above inclusion relation are strict.

(3.6) <u>Example:</u> Let X be R (the set of all reals). L_1 is generated by $\{(a,b)$ and $[2,a)$ where $a,b \in R\}$ and L_2 is the usual topology. Then $[1,2]$ is a compact subset in L_2 but is not p-compact which can be seen by considering $L_2([2,3))$. Now (X, L_1, L_2) is p-T_2 and each L_i is Hausdorff for $i = 1,2$ and has an L_2-compact subset which is not p-compact. If (Y, S_1, S_2) is any bitopological space then $\widetilde{K}([1,2],V) \in \kappa_2 - \psi_2$ for $V \in S_2$.

(3.7) <u>Theorem:</u> If (X, L_1, L_2) is p-compact and L_i is Hausdorff, $i = 1,2$, then

$$\psi_i = \kappa_i , \quad \text{for} \quad i = 1,2 .$$

<u>Proof:</u> By the previous lemma we have only to show that $\kappa_i \subset \psi_i$. Consider $\widetilde{K}(C,U)$ where $U \in S_i$ and C is L_i-compact. Then C is L_j-closed by (2.13) (recall that X is p-Hausdorff) . Thus C is p-compact by Corollary (2.11) which implies that $\widetilde{K}(C,U) \in \psi_i$.

Since every singleton is a p-compact we have the following:

(3.8) <u>Theorem:</u> The pairwise pointwise convergent topology, denoted by

p- Ptopology, is contained in the p-K topology.

Thus we have the following relation:

$P_i \subset \psi_i \subset \kappa_i$ (where P_i is generated by the subbase $\{(X,U) = \{f \in Y^X: f(X) \in U \in S_i\}\}$) provided L_i is Hausdorff, $i = 1,2$.

(3.9) <u>Definition:</u> Let (Y, S_1, S_2) be a bitopological space generated by a p-q metric p and its conjugate q . Then $\{f_n\}_1^\infty \subset C_{S_{12}}^{L_{12}}$ is said to <u>p-converge to</u> $f \in Y^X$ <u>uniformly on p-compact sets</u> iff, for each p-compact set $C \subset X$ and $\varepsilon > 0$, there exist $N = N(C, \varepsilon)$ such that

$$p(f(c_i), f_n(c_i)) \vee q(f(c_i), f_n(c_i)) < \varepsilon \quad \text{for all } n \geq N \text{ and for all}$$

$c_i \in C$.

The following is a generalization of Arens' theorem [1] on sequential convergence in the K topology.

(3.10) <u>Theorem:</u> Let (Y, S_1, S_2) be quasi metric and let S_i be Hausdorff for $i = 1,2$. Then $\{f_n\}_1^\infty \subset C_{S_{12}}^{L_{12}}$ p-converges to $f \in C_{S_{12}}^{L_{12}}$ uniformly on every p-compact subset iff $f_n \to f$ in the p-K topology of $C_{S_{12}}^{L_{12}}$.

<u>Proof:</u> Assume $\{f_n\}_1^\infty$ p-converge to f uniformly on every p-compact subset of X . Let $f \in \tilde{K}(C,V)$ where $V \in S_1$ (we have an equivalent result for $V \in S_2$) , which implies $f(C) \subset V$. Since C is p-compact f(C) is p-compact and f(C) is closed and compact in S_i , $i = 1,2$,

by (2.12) and the remark following (3.3) . Therefore we have $\varepsilon_i > 0$, $i = 1,2$, such that

$$p(f(C), V^c) = \varepsilon_1$$

$$q(f(C), V^c) \leq \varepsilon_2 .$$

Let $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ and $N = N(C,\varepsilon)$. Then for all $n \geq N$, we have

$$p(f_n(c_i), f(c_i)) \vee q(f_n(c_i), f(c_i)) < \varepsilon$$

for all $c_i \in C$, which implies

$$f_n \in \widetilde{K}(C,V) \qquad \text{for all} \quad n \geq N .$$

Hence $f_n \to f$ in the p-K topology.

Conversely, for a p-compact subset K in X , $f(K)$ is a p-compact subset in Y for any $f \in C_{S_{12}}^{L_{12}}$. Since Y is a Hausdorff space, $f(K)$ is S_i-compact by the remark following (3.3) $i = 1,2$. Now, for each $c_i \in K$, consider $P_i(f(c_i),\frac{\varepsilon}{2}) = \{y : p(f(c_i),y) < \frac{\varepsilon}{2}\}$ and $Q_i(f(c_i),\frac{\varepsilon}{2}) = \{y : q(f(c_i),y) < \frac{\varepsilon}{2}\}$.

Since (Y, S_1, S_2) is a quasi metric space, it is a p-regular space by (1.9). Therefore there exist $U_i \subset E_i \subset P_i(f(c_i),\frac{\varepsilon}{2})$ and $V_i \subset D_i \subset Q_i(f(c_i),\frac{\varepsilon}{2})$ where $f(c_i) \in U_i \in S_1$ and E_i is an S_2-closed set and $f(c_i) \in V_i \in S_2$ and D_i is an S_1-closed set.

By (3.4)

$$K_1 = K \cap f^{-1}(E_1)$$

$$\vdots \quad \vdots \quad \vdots$$

$$K_n = K \cap f^{-1}(E_n)$$

$$K_1' = K \cap f^{-1}(D_1)$$

$$\vdots \quad \vdots \quad \vdots$$

$$K_m' = K \cap f^{-1}(D_m)$$

are p-compact subsets in (X, L_1, L_2) and the compactness os $f(K)$ allows us to cover $f(K)$ with a finite number of S_1 and S_2 open sets $\{U_i\}_1^n$ and $\{V_i\}_1^m$ respectively. Therefore

$$K = K_1 \cup \ \ldots \ \cup K_n$$

$$= K_1' \cup \ \ldots \ \cup K_m'$$

where K_i , $i = 1,\ldots,n$ and K_i' , $i = 1,\ldots,m$, are p-compact subsets of X .

Consider
$$U = \tilde{K}(K_1,P_1) \cap \ \ldots \ \cap \tilde{K}(K_n,P_n)$$

$$V = \tilde{K}(K_1',Q_1) \cap \ \ldots \ \cap \tilde{K}(K_m',Q_m)$$

where P_i is a p-sphere of radius $\frac{\varepsilon}{2}$ about $f(c_i)$ and Q_j is a q-sphere of radius $\frac{\varepsilon}{2}$ about $f(c_j)$. Then U and V are p-K neighborhoods of f with respect to ψ_1 and ψ_2 , respectively, which implies there exists N such that for $n \geq N$, $f_n \in U \cap V$. If $x \in K$ then $x \in K_i$ for some i and if

$$f_n(x) \in P_i \cap Q_i$$

then

$$p(f_n(x),f(x) \vee q(f_n(x),f(x)) < \varepsilon$$

(3.11) Example: We have a nontrivial (i.e. nonmetric) quasi metric bitopological space (Y, S_1, S_2) such that each S_i is Hausdorff, for $i = 1,2$, which is defined as follows:

$d(x,y) = \min (1, |x-y|)$ for all $x,y \in R$ (the set of all reals)

$$p(x,y) = \begin{cases} d(x,y) & \text{if} \quad x \leq y \\ 1 & \text{if} \quad x > y \end{cases}$$

$$q(x,y) = \begin{cases} d(x,y) & \text{if} \quad x \geq y \\ 1 & \text{if} \quad x < y \end{cases}$$

The topology S_1 determined by p is the topology for R which has a base the family of all half open intervals of the form $[a,b) = \{x: a \leq x < b\}$, while S_2 is determined by q with base all half open intervals of the form $(a,b]$.

(3.12) Example: If, in the hypothesis of (3.10), we assume a p-q metric instead of quasi metric T_2 space, then the statement is not true. As a counter example consider $X = Y = R$ (the set of all reals). Now $L_1 = S_1$ is the topology generated by the base consisting of all sets of the form $(-\infty,a) = \{y: y < a\}$, while $L_2 = S_2$ is generated by the base consisting of all sets $(a,\infty) = \{y: a < y\}$. Then (Y, S_1, S_2) is a p-q metric space generated by p: $p(x,y) = 0$ if $y \leq x$ and $p(x,y) = |x-y|$ if $y > x$ and q, the conjugate of p.

Define the sequence $\{f_n\}_1^\infty$ by

$$f_n(x) = x + \frac{1}{n} \quad \text{for all } x \in X .$$

Then $\{f_n\}_1^\infty$ p-converges uniformly to $f = 1$ (identity map) on every p-compact subset of X. However,

$$f \in \widetilde{K}((-\infty,a), (-\infty,a)) \in \psi_1$$

while, for any n, $f_n \notin K((-\infty,a), (-\infty,a))$ when $a \neq \infty$ i.e.

$$f_n \nrightarrow f$$

in the p-K topology.

(3.13) <u>Theorem:</u> $(C_{S_{12}}^{L_{12}}, \psi_1, \psi_2)$ is p-T_o, p-T_1, or p-T_2 whenever (Y, S_1, S_2) is p-T_o, p-T_1, or p-T_2, respectively. (For definition p-T_i refer to [15])

<u>Proof:</u> Since p-T_o, p-T_1 can be shown analogously, we show only the case of p-T_2. Let $f,g \in C_{L_{12}}^{S_{12}}$ and $f \neq g$. Then there exists an $x \in X$ such that $f(x) \neq g(x)$. Since (Y, S_1, S_2) is p-Hausdorff, there exist $U \in L_i$ and $V \in L_j$ such that

$$g(x) \in V , \quad f(x) \in U \text{ and } U \cap V = \emptyset .$$

Thus $\widetilde{K}(x,U)$ and $\widetilde{K}(x,V)$ are disjoint neighborhoods of f and g in ψ_i and ψ_j, respectively, $i \neq j$, $i,j = 1,2$.

(3.14) <u>Theorem</u>: $(C_{S_{12}}^{L_{12}}, \psi_1, \psi_2)$ is p-regular whenever (X, L_1, L_2) is p-compact and L_i-Hausdorff with respect to $i = 1,2$, and (Y, S_1, S_2) is p-regular.

<u>Proof</u>: By Theorem (3.7) $\psi_i = \kappa_i$, $i = 1,2$. Consider

$$f \in \tilde{K}(K,U) \in \psi_i$$

where K is L_i-compact and $U \in S_i$. Now $f(K)$ is S_i-compact. For each $x_i \in f(K)$ we have

$$x_i \in V_{x_i} \subset C_{x_i} \subset U$$

where V_{x_i} is S_i-open C_{x_i} is S_j-closed. Since $f(K)$ is compact in S_i we have

$$f(K) \subset \bigcup_1^n V_{x_i} = V \in S_i .$$

Let $\bigcup_i^n C_{x_i} = C$. Then C is S_j-closed with $C \subset U$. Thus

$$f(K) \subset V \subset C \subset U$$

Suppose $g \in C_{S_{12}}^{L_{12}}$ and $g \notin \tilde{K}(K,U)$. Then there exists $x \in K$ such that $g(x) \notin U$. Consider $\tilde{K}(x,C^c)$. Since x is p-compact and $C^c \in S_j$, then $\tilde{K}(x,C^c) \in \psi_j$ with $g \in K(x,C^c)$ but $\tilde{K}(K,V) \subset \tilde{K}(K,U)$ and $\tilde{K}(K,V) \in \psi_i$, and

$$\tilde{K}(x,C^c) \cap \tilde{K}(K,V) = \emptyset .$$

Hence if $g \notin \tilde{K}(K,U)$ then $g \notin \overline{\tilde{K}(K,V)}^j$ and the result is proved.

(3.15) <u>Corollary:</u> With topology κ_i, the space $C_{S_{12}}^{L_{12}}$ is p-T_o, p-T_1, or p-T_2 whenever (Y, S_1, S_2) is p-T_o, p-T_1 or p-T_2, respectively.

(3.16) <u>Corollary:</u> With the K topology, the space C_L^S is T_o, T_1 or T_2 whenever (Y, S) is T_o, T_1 or T_2, respectively.

(3.17) <u>Corollary:</u> With the K topology, the space $C_{S_{12}}^{L_{12}}$ is p-regular whenever (Y, S_1, S_2) is p-regular.

(3.18) <u>Corollary:</u> With the K topology, the space C_S^L is regular whenever (Y, S) is regular.

CHAPTER 4

(12)-CONNECTED OPEN TOPOLOGY

As a generalization of the connected open topology which was initiated by Irudayanathan and Naimpally [9], we consider the (12)-connected open topology which is denoted by C^{*}_{12}. First, we discuss some definitions, theorems and notations which will be used in the sequel.

(4.1) <u>Definition:</u> In a bitopological space (X, L_1, L_2) a pair (A,B) with $A,B \subset X$ is said to be <u>(12)-separated</u> iff

$$\overline{A} \cap B = A \cap \overline{\overline{B}} = \emptyset$$

where $\overline{A}$ is the L_1-closure of A and $\overline{\overline{B}}$ is the L_2-closure of B.

<u>Remark:</u> If $L_1 \subset L_2$ then every (12)-separated pair is L_2-separated.

(4.2) <u>Lemma:</u> In a bitopological space (X, L_1, L_2), (A,B) is (12)-separated iff there exist $W_p \in L_1$ and $W_q \in L_2$ such that $A \subset W_q$ and $B \subset W_p$ with $W_q \cap B = \emptyset$ and $W_p \cap A = \emptyset$.

<u>Proof:</u> Assume (A,B) is (12)-separated. Then $\overline{A} \cap B = A \cap \overline{\overline{B}} = \emptyset$ and $\overline{A} \cap B = \emptyset$ implies $(\overline{A})^c \supset B$ and $(\overline{A})^c \cap A = \emptyset$. Let $(\overline{A})^c = W_p$. Similarly $W_q = (\overline{\overline{B}})^c$.

Conversely, assume $W_q \in L_2$ such that $A \subset W_q$ and $W_q \cap B = \emptyset$. Then $W_q^c \supset B$. But W_q^c is L_2-closed which implies $W_q^c \supset \bar{\bar{B}}$, while $W_q \supset A$. This $A^c \supset W_q^c$ and $A^c \supset \bar{\bar{B}}$. Therefore

$$A \cap \bar{\bar{B}} = \emptyset .$$

Similarly

$$\bar{A} \cap B = \emptyset .$$

(4.3) <u>Definition:</u> A bitopological space (X, L_1, L_2) is <u>(12)-connected</u> iff X is not the union of two non-void (12)-separated subsets. Similarly we can define (21)-connected and extend the definition to p-connected.

(4.4) <u>Definition:</u> (X, L_1, L_2) is said to be <u>p-completely normal</u> iff for every (12)-separated pair (A,B) there exist L_2-open set $U \supset A$ and L_1-open set $V \supset B$ such that $U \cap V = \emptyset$.

With the above definition we have

(4.5) <u>Theorem:</u> (X, L_1, L_2) is p-completely normal iff every subset of X is p-normal.

<u>Proof:</u> Suppose X is p-completely normal and $Y \subset X$. Let F_1 and F_2 be disjoint closed (relative to Y) in L_1 and L_2, respectively. Then

$$F_1 \cap \bar{\bar{F}}_2 = \bar{F}_1^{L_{Y_1}} \cap \bar{\bar{F}}_2 = (Y \cap \bar{F}_1) \cap \bar{\bar{F}}_2 = \bar{F}_1^{L_{Y_1}} \cap \bar{F}_2^{L_{Y_2}} = F_1 \cap F_2 = \emptyset$$

where $\overline{F}$ is the L_1-closure of F and $\overline{F}^{L_{Y_i}}$ denotes the L_{Y_i}-closure of F. Similarly, we can show $\overline{F}_1 \cap F_2 = \emptyset$ which implies (F_1, F_2) is a (12)-separated pair of X. By p-complete normality there exist disjoint sets(L_1-open) G_1 and (L_2-open) G_2 containing F_2 and F_1, respectively. Then $Y \cap G_1$ and $Y \cap G_2$ are disjoint L_{Y_1}, L_{Y_2} open sets of Y which contain F_2 and F_1, so that Y is a p-normal.

Conversely, let (A,B) be a (12)-separated pair, i.e.

$$(\overline{A} \cap B) \cup (A \cap \overline{\overline{B}}) = \emptyset .$$

Let $Y = (\overline{A} \cap \overline{\overline{B}})^c$. Then (Y, L_{Y_1}, L_{Y_2}) is p-normal by assumption. Since

$$Y \cap \overline{A} = (\overline{A}^c \cup \overline{\overline{B}}^c) \cap \overline{A} = \overline{\overline{B}}^c \cap \overline{A}$$

$$Y \cap \overline{\overline{B}} = (\overline{A}^c \cup \overline{\overline{B}}^c) \cap \overline{\overline{B}} = \overline{A}^c \cap \overline{\overline{B}}$$

$Y \cap \overline{A}$ and $Y \cap \overline{\overline{B}}$ are disjoint L_{Y_1} and L_{Y_2}-closed sets, respectively. Therefore, there exist $U \cap Y = U_Y \in L_{Y_1}$ and $V \cap Y = V_Y \in L_{Y_2}$ such that $(Y \cap \overline{\overline{B}}) \subset U_Y$ and $(Y \cap \overline{A}) \subset V_Y$, where $U_Y \cap V_Y = \emptyset$. But $U \cap (\overline{A} \cap \overline{\overline{B}})^c = U \cap (\overline{A}^c \cup \overline{\overline{B}}^c)$

$$U_Y = U \cap Y = (U \cap \overline{\overline{B}}^c) \cup (U \cap \overline{A}^c) \quad \text{where} \quad U \in L_1$$

$$V_Y = V \cap Y = (V \cap \overline{\overline{B}}^c) \cup (V \cap \overline{A}^c) \quad \text{where} \quad V \in L_2$$

Since $(U \cap \overline{\overline{B}}^c) \cap \overline{\overline{B}} = \emptyset$,

$$U_Y \supset (Y \cap \overline{\overline{B}}) \quad \text{implies} \quad (U \cap \overline{A}^c) \supset (Y \cap \overline{\overline{B}})$$

and

$$V_Y \supset (Y \cap \bar{A}) \text{ implies } (V \cap \bar{\bar{B}}^c) \supset (Y \cap \bar{A}) .$$

Since $U \in L_1$ and $\bar{A}^c \in L_1$ $U' = U \cap \bar{A}^c \in L_1$ and $U_Y \supset U'$. Similarly $V' = V \cap \bar{\bar{B}}^c \in L_2$ and $V_Y \supset V'$. Since $U_Y \cap V_Y = \emptyset$, this implies $U' \cap V' = \emptyset$. Consider

$$Y \cap \bar{\bar{B}} = (\bar{A}^c \cup \bar{\bar{B}}^c) \cap \bar{\bar{B}} = \bar{A}^c \cap \bar{\bar{B}} .$$

But $\bar{A} \cap B = \emptyset$ so that $\bar{A}^c \supset B$. Therefore

$$\bar{A}^c \cap \bar{\bar{B}} \supset B \text{ and } U' = U \cap \bar{A}^c \supset (Y \cap \bar{\bar{B}}) = \bar{A} \cap \bar{\bar{B}} \supset B .$$

Similarly,

$$V' \supset (\bar{\bar{B}}^c \cap \bar{A}) \supset A .$$

(4.6) <u>Definition:</u> (Y,U,L) is said to be a <u>semi continuous bitopological space</u> iff U is generated by $\{(-\infty, a)\}$ and L is generated by $\{(a,\infty)\}$ where Y is the space of reals and a is any real number.

(4.7) <u>Lemma:</u> Every subspace of a p-q metric space (X, P, Q) is a p-q metric space.

(4.8) <u>Theorem:</u> A p-q metric space (X, P, Q) is p-completely normal.

<u>Proof:</u> By Theorem (1.9) a p-q metric space is p-normal, and by the above lemma every subspace of a p-q metric space is p-q metric

space also. Therefore every subspace is p-normal, and, so by Theorem (4.5) we have p-complete normality for X.

(4.9) <u>Lemma:</u> If in a bitopological space (X, L_1, L_2) a subset A is (12)-connected, then, for any $f \in C_{S_{12}}^{L_{12}}$, $f(A)$ is (12)-connected in (Y, S_1, S_2).

<u>Proof:</u> Assume $f(A) = U \cup V$ where U, V are non-empty and $\overline{U}^{S_1} \cap V = U \cap \overline{V}^{S_2} = \emptyset$ where $\overline{U}^{S_i}$ denote S_i-closure of U. By (4.2) there exist $W_p \in S_1$ and $W_q \in S_2$ such that $U \subset W_q$ and $V \subset W_p$ where $W_q \cap V = W_p \cap U = \emptyset$ which implies $f^{-1}(W_p) \in L_1$ and $f^{-1}(W_q) \in L_2$ with

$$f^{-1}(W_p) \cap f^{-1}(U) = f^{-1}(W_q) \cap f^{-1}(V) = \emptyset$$

where $f^{-1}(W_p) \supset f^{-1}(V)$ and $f^{-1}(W_q) \supset f^{-1}(U)$. Therefore $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = A$ is (12)-separated which contradicts the assumption.

(4.10) <u>Definition:</u> For each (12)-connected subset K of (X, L_1, L_2) and $U \in S_1$, $V \in S_2$ in (Y, S_1, S_2), let

$$\tilde{C}(K,U,V) = \{f \in Y^X : f(K) \subset U \cup V, f(K) \cap U \neq \emptyset \neq f(K) \cap V, U \in S_1 \text{ and } V \in S_2\}.$$

The topology C_{12}^{*} generated by the subbase $\{\tilde{C}(K,U,V)\}$ is called a <u>(12)-connected open topology</u> and is denoted by (Y^X, C_{12}^{*}).

We use the following notations for further discussion:

(1) $C^{-2}\,{}^{L_{12}}_{S_{12}}$ = the set of all p-connected mappings (i.e. mappings which preserve p-connectedness)

$$f:(X, L_1, L_2) \to (Y, S_1, S_2)$$

(2) $C^{-2}\,{}^{L}_{S}$ = the set of all connected mappings

$$f:(X, L), \to (Y, S)$$

(3) C^{-2}_{12} = the set of all (12)-connected mappings (i.e. mappings which preserve (12)-connectedness)

$$f:(X, L_1, L_2) \to (Y, S_1, S_2)$$

(4.11) <u>Theorem</u>: If (Y, S_1, S_2) is p-completely normal, then C^{-2}_{12} is closed in (Y^X, C^{*}_{12}) .

<u>Proof</u>: Suppose $f \in Y^X$ is a limit point of C^{-2}_{12} but $f \notin C^{-2}_{12}$. Then there exists a (12)-connected set $K \subset X$ such that $f(K)$ is not a (12)-connected subset of (Y, S_1, S_2) . Since Y is p-completely normal, there exist non-empty disjoint sets S_1-open U and S_2-open V such that

$$f(K) \subset U \cup V \ , \quad f(K) \cap U \neq \emptyset \neq f(K) \cap V \ .$$

Since f is a limit point of C^{-2}_{12} , there exists a $g \in C^{-2}_{12}$ such that $g \in \widetilde{C}(K,U,V)$ where $\widetilde{C}(K,U,V)$ is an open neighborhood of f in C^{*}_{12} . So $g(K) \subset U \cup V$ with $g(K) \cap U \neq \emptyset \neq g(K) \cap V$. But $U \cap V = \emptyset$

which is a contradiction. Therefore $f \in C_{12}^{-2}$ and the result is proved.

(4.12) <u>Theorem</u>: If X is an interval with the usual topology and (Y, S_1, S_2) is a semi continuous bitopological space, then the pointwise convergence (or P_i, $i = 1,2$) topology is strictly smaller than C_{12}^{*}.

<u>Proof</u>: By (4.9) $C_{S_{12}}^{L_{12}} \subset C_{12}^{-2}$. Now $C_{S_{12}}^{L_{12}}$ is dense in $P_i (i = 1,2)$ respectively, but (Y, S_1, S_2) is a p-completely normal space by virtue of Theorem (4.8). By Theorem(4.11), C_{12}^{-2} is closed in C_{12}^{*}. Hence the theorem follows.

(4.13) <u>Definition</u>: (X, L_1, L_2) is said to be <u>bi-locally connected</u> iff (X, L_1) and (X, L_2) are locally connected.

With the above definition we can construct a topology which is comparable with p-compact open topology on $C_{S_{12}}^{L_{12}}$. However, we will not discuss it here and the relation between p-compact open topology and p-connected open topology is considered in the next chapter.

CHAPTER 5

p-CONNECTED OPEN TOPOLOGY

In Chapter 4 we discussed one of the generalized forms of connected open topology. Now, in this chapter we consider another generalization in the bitopological sense and discuss mainly the separation axioms and the comparison with other bitopological function spaces.

(5.1) <u>Definition:</u> p-connected open topology on Y^X is defined with the subbase

$$C_i(K,U,V) = \{f \in Y^X : f(K) \subset U \cup V,\ f(K) \cap U \neq \emptyset \neq f(K) \cap V\}$$

where $U,V \in S_i$, $i = 1,2,$ and K is a p-connected subset of X , and if Γ_i is generated by $\{C_i(K,U,V))\}$ then

$$(Y^X, \Gamma_1, \Gamma_2)$$

is said to be a <u>p-connected open bitopological function space.</u>

(5.2) <u>Theorem:</u> $(Y^X, \Gamma_1, \Gamma_2)$ is p-T_o , p-T_1 , or p-T_2 whenever Y is p-T_o , p-T_1 , or p-T_2 , respectively.

<u>Proof:</u> We give a proof for the case Y is p-T_2 . If $f,g \in Y^X$ and $f \neq g$ then there exists an $x \in X$ such that $f(x) \neq g(x)$. Since Y is p-T_2 , we have $U \in S_1$ and $V \in S_2$ such that $f(x) \in U$ and $g(x) \in V$

with $U \cap V = \emptyset$. Since a singleton is a p-connected set $C_1(x, U, U) \in \Gamma_1$ $C_2(x, V, V) \in \Gamma_2$ where

$$C_1(x, U, U) \cap C_2(x, V, V) = \emptyset .$$

(5.3) Theorem: $(Y^X, \Gamma_1, \Gamma_2)$ is p-regular whenever (Y, S_1, S_2) is p-regular.

Proof: Consider a Γ_1-nbhd of f, $U(f) = \bigcap_1^n C_1(K_i, U_i, V_i) \in \Gamma_1$. since Y is p-regular there exist $U_i', V_i' \in S_1$ and S_2-closed sets D_i and E_i such that

$$U_i' \subset D_i \subset U_i$$

$$V_i' \subset E_i' \subset V_i'$$

Suppose $g \notin U(f)$ which implies there exists an i such that

(1) $g(K_i) \not\subset U_i \cup V_i$ or

(2) either $g(K_i) \subset U_i \cup V_i$ and $g(K_i) \cap U_i = \emptyset$ or $g(K_i) \cap V_i = \emptyset$

Without loss of generality we can assume, there exists an i such that $x \in K_i$ and $g(x) \notin U_i$, which implies $g(x) \notin D_i$, i.e. $g(x) \in D_i^c \in S_2$. Consider $C_2(x, D_i^c, D_i^c) \in \Gamma_2$ and let $V(f) = \bigcap_1^n C_1(K_i, U_i', V_i') \in \Gamma_1$.

$$V(f) \cap C_2(x, D_i^c, D_i^c) = \emptyset .$$

Hence $f \in V(f) \subset \overline{V(f)}^{\Gamma_2} \subset U(f)$.

Further, assuming $U(f) \in \Gamma_2$, we have an equivalent result. Therefore p-regularity is proved.

(5.4) <u>Theorem:</u> If (Y, S_1, S_2) is a bitopological space then

$$P_i \subset \Gamma_i, \quad i = 1,2 .$$

<u>Proof:</u> A subbase for the P_i is the set (x,U) where

$$(x,U) = \{ f \in Y^X : f(x) \in U \in S_i , \quad i = 1,2 \} .$$

But $(x,U) = C_i(x ,U,U) \in \Gamma_i$, $i = 1,2$.

(5.5) <u>Theorem:</u> If a bitopological space (X, L_1, L_2) is both Hausdorff and bilocally connected then $\kappa_i \subset \Gamma_i$, for $i = 1,2$.

<u>Proof:</u> Let $\tilde{K}(K,U) \in \kappa_i$. Then $h \in \tilde{K}(K,U)$ implies $h(K) \subset U \in S_i$. Since K is L_i-compact

$$K \subset (\bigcup_1^n W_t)$$

where $W_t \in L_i$, $W_t \subset h^{-1}(U)$, $t = 1,2,\dots n$, where W_t is non-empty connected set in L_i;

then
$$\bigcap_1^n (W_t, U, U) \in \Gamma_i \quad \text{and}$$

$$\bigcap_1^n (W_t, U, U) \subset \tilde{K}(K\ U) .$$

(5.7) <u>Corollary:</u> If a bitopological space (X, L_1, L_2) is p-Hausdorff and both L_i is Hausdorff then

$$\psi_i \subset \Gamma_i \text{ , for } i = 1,2 \text{ .}$$

<u>Proof:</u> See (3.5) and (5.6) .

CHAPTER 6

BIGRAPH TOPOLOGY

The graph topology was initiated by Naimpally [16] for the purpose of studying almost continuous functions. In this chapter we study it in the bitopological sense.

(6.1) Definition: $f \in Y^X$ is said to be almost continuous in $(X, L) \times (Y, S)$ iff for each open set $U \in L \times S$ containing

$$C(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$$

there exists a $g \in C_S^L$ such that $C(g) \subset U$.

Notations: (1) A_S^L will denote the set of all almost continuous functions in $L \times S$.

(2) For each $U \in L \times S$, let $G_U = \{f \in Y^X : C(f) \subset U\}$.

(6.2) Definition: The topology $G_{L \times S}$ on Y^X generated by $\{G_U : U \in L \times S\}$ is called graph topology.

We extend the above definition to bitopological space.

(6.3) Definition: Let (X, L_1, L_2) and (Y, S_1, S_2) be bitopological spaces, and let

$$G_{L_\ell \times S} = \{G_U : U \in L_\ell \times S\}, \quad \ell = 1,2 .$$

Then

$$\tilde{G}_S^{L_{12}} = \{Y^X, G_{L_1 \times S}, G_{L_2 \times S}\}$$

is said to be a <u>bigraph topology</u> on Y^X generated by L_1 and L_2, and

$$\tilde{G}_{S_{12}}^{L} = \{Y^X, G_{L \times S_1}, G_{L \times S_2}\}$$

is said to be a <u>bigraph topology</u> on Y^X generated by S_1 and S_2.

We will frequently make use of the following:

(6.4) <u>Theorem:</u> (Naimpally [16]). If Y contains at least two points, then the following are equivalent:

(1) (X, L) and (Y, S) are T_1 spaces

(2) $(Y^X, G_{L \times S})$ is T_1

(3) $(A_S^L, G_{L \times S})$ is T_1

As an extension of the above theorem we have:

(6.5) <u>Theorem:</u> (1) (Y, S_1, S_2) has at least two points, then

$$\tilde{G}_{S_{12}}^{L} = (Y^X, G_{L \times S_1}, G_{L \times S_2})$$

is p-Hausdorff iff (Y, S_1, S_2) is p-Hausdorff and (X, L) is T_1. If

also, (X, L_1, L_2) has at least two points then

$$(2) \quad \tilde{G}_S^{L_{12}} = (Y^X, G_{L_1 \times S}, G_{L_2 \times S})$$

is p-Hausdorff iff (X, L_1, L_2) is p-Hausdorff and (Y, S) is T_1 .

<u>Proof:</u> Assume $\tilde{G}^L_{S_{12}}$ is p-Hausdorff but (Y, S_1, S_2) is not p-Hausdorff. Let p,q Y , $p \neq q$. Then every open set $O_p \in S_1$ intersects every $U_q \in S_2$ where $p \in O_p$ and $q \in U_q$. Define $f,g \in Y^X$ such that $f(x) = p$ and $g(x) = q$ for all $x \in X$. Then every open set containing $C(f)$ in $G_{L \times S_1}$ intersects every open set in $G_{L \times S_2}$ containing $C(g)$ which is a contradiction. Assume (X, L) is not T_1 . By Theorem (6.4) $\tilde{G}^L_{S_1}$ and $\tilde{G}^L_{S_2}$ are not T_1 . Since a p-Hausdorff space is T_1, $\tilde{G}^L_{S_{12}}$ is not p-Hausdorff.

Conversely, assume that (Y, S_1, S_2) is p-Hausdorff and (X, L) is T_1 . Now if f,g Y^X and $f \neq g$, then there exists $a \in X$ such that $f(a) \neq g(a)$. Since (Y, S_1, S_2) is p-Hausdorff there exist $U \in S_1$ and $V \in S_2$ such that $f(a) \in U$ and $g(a) \in V$ with $U \cap V = \emptyset$ which implies

$$W_1 = (X \times U) \cup ((X - a) \times Y) \in S_1 \times L$$

$$W_2 = (X \times V) \cup ((X - a) \times Y) \in S_2 \times L$$

and $G_{W_1} \cap G_{W_2} = \emptyset$ with $C(f) \in W_1$ and $C(g) \in W_2$. Therefore $\tilde{G}^L_{S_{12}}$ is p-Hausdorff.

To prove part (2) first assume (X, L_1, L_2) is not P-Hausdorff, i.e. for $a, b \in X$, $a \neq b$ every open set containing a in L_1 intersects every open set in L_2 that contains b. Let $p \neq q$, and $p, q \in Y$. We define $f, g \in Y^X$ such that $f(b) = q$, $g(b) = p$, $f(x) = g(x) = p$ for all $x \in X$, $x \neq b$. Every open set in $G_{L_1} \times S$ which contains $C(f)$, intersets with every open set in $G_{L_2} \times S$ which contains $C(g)$ which is a contradiction. Further, if S is not T_1 then $\tilde{G}_S^{L_i}$ $i = 1,2$ is not T_1 by (6.4). Therefore $\tilde{G}_S^{L_{12}}$ cannot be p-Hausdorff which is a contradiction.

Conversely, assume that (X, L_1, L_2) is p-Hausdorff and (Y, S) is T_1. If $f, g \in Y^X$, $f \neq g$, then there exists a $p \in Y$ such that $a = f^{-1}(p) \neq g^{-1}(p) = b$. Since X is p-Hausdorff there exists an $U_a \in L_1$ and $V_b \in L_2$ with $a \in U_a$ and $b \in V_b$ where $U_a \cap V_b = \emptyset$. Consider

$$W_1 = (Y \times U_a) \cup ((Y-p) \times X \in S \times L_1$$

$$W_2 = (Y \times V_b) \cup ((Y-p) \times X \in S \times L_2$$

then $G_{W_1} \cap G_{W_2} = \emptyset$. Furthermore $C(f) \subset W_1$ and $C(g) \subset W_2$. Therefore

$$\tilde{G}_S^{L_{12}} = (Y^X, G_{L_1} \times S, G_{L_2} \times S)$$

is p-Hausdorff.

Since f and g in the above throrem can be chosen to be almost continuous functions we obtain the following:

(6.6) Corollary: If (Y, S_1, S_2) has at least two points, then $A^{L}_{S_{12}}$ with graph topology is p-Hausdorff iff (Y, S_1, S_2) is p-Hausdorff and (X, L) is T_1. If, also, (X, S_1, S_2) has at least two points, then $A^{L_{12}}_{S}$ is p-Hausdorff iff (X, L_1, L_2) is p-Hausdorff and (Y, S) is T_1.

We now consider to D = the set of all first kind of discontinuous functions in the following way. For $f \in D$ let f^+ denote that function $f^+(x) = \lim_{y\, x^+} f(y)$ for all $y \in R$ (reals). Similarly $f^-(x)$ can be defined on R.

(6.7) Definition: (Almost first kind of discontinuous functions (A.F.K.D)) $f \in \{A.F.K.D.\}$ iff for any open set $U_\ell \in L_i \times S$, $i = 1,2$ and $\ell = 1,2$, such that $C(f) \subset U_1$, $C(f) \subset U_2$ there exists $g \in D$ such that $C(g^+) \subset U_1$, $C(g^-) \subset U_2$ where $Y = R$ (reals) with the usual topology and $X = R$ with $L_1 = \{[a,b) : a,b \in R\}$ while $L_2 = \{(a,b] : a,b \in R\}$.

We construct an example which is A.F.K.D.

(6.8) Example: f is defined as follows:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } x > 0 \\ 2 + \sin \frac{1}{x} & \text{for } x < 0 \\ 1 & \text{for } x = 1 \end{cases}$$

where the topologies are that defined in Definition (6.7)

Also we have the following.

(6.9) <u>Corollary</u>: The family of A.F.K.I. with the topologies in (6.7) is closed in $G_S^{L_{12}}$.

Now we discuss the relationship between different function space topologies.

(6.10) <u>Theorem</u>: If (X, L_1, L_2) is p-Hausdorff, then p-P topology is contained in the bigraph topology generated by L_1 and L_2 .

<u>Proof</u>: Since $(X; L_1, L_2)$ is p-Hausdorff, a singleton is closed in L_i , $i = 1,2$. Therefore

$$W = (X \times U) \cup ((X - a) \times Y)$$

is open in $L_i \times S$, $i = 1,2$, and

$$G_W = [a,U] = \{f \in Y^X : f(a) \subset U\} .$$

Therefore $P_i \subset G_{L_i} \times S$, $i = 1,2$.

(6.11) <u>Theorem</u>: If (X, L_1, L_2) is p-Hausdorff, then the graph topology $G_{L_j \times S_i}$ contains compact open topology κ_i .

<u>Proof</u>: Consider $f \in \widetilde{K}(K,U) = \{ f \in Y^X : f(K) \subset U \in S_i$, K is L_i-compact$\}$. Since X is p-Hausdorff, K is L_j-closed. Let

$$V = (X \times U) \cup ((X - K) \times Y) \in L_j \times S_i$$

Then $G_V = \widetilde{K}(K,U)$. Therefore, any open set in κ_i is an element of

$G_{L_j} \times S_i$.

(6.12) <u>Corollary:</u> If (X, L_1, L_2) is p-Hausdorff and Hausdorff with respect to L_i , $i = 1,2$, then

$$\psi_i \subset G_{L_i} \times S_i$$

<u>Proof:</u> By (2.12) a p-compact set is L_i-closed, $i = 1,2$. Apply a method similar to (6.11), we obtain the result required.

(6.13) <u>Corollary:</u> If (X, L_1, L_2) is p-Hausdorff and L_i-Hausdorff, $i = 1,2$, then we have the following relation:

$$P_i \subset \psi_i \subset \kappa_i \subset G_{L_i} \times S_i \quad i = 1,2 .$$

<u>Proof:</u> See Lemmas (3.5), (6.10) and (6.11).

If a topological space (X, L) is compact and Hausdorff then

$$\kappa_i = G_{L_i} \times S_i \quad \text{(See [16])} .$$

Therefore, we have the following.

(6.14) <u>Corollary:</u> If (X, L_1, L_2) is p-compact and p-Hausdorff and L_i-Hausdorff $i = 1,2$, then the bigraph topology coincides with the p-compact open topology for $C_{S_{12}}^{L_{12}}$.

CHAPTER 7

ALMOST CONVERGENT TOPOLOGY

The study in function space topologies mainly has been investigated in the space of continuous functions (see [16]). Recently Kolmogorov [12], Prokhorov [20], and Skorokhod [23] discussed topologies on the space of all discontinuous functions of the first kind in connection with a problem in probability theory. In the theory of probability, if the independent variable t is considered to be the time, then it is impossible to assume the existence of an instrument which will measure time exactly whence a comparatively weaker topology is considered (see [23]). In this chapter for the above mentioned purpose almost convergent topology is considered and in the end of this chapter one shows Skorokhod M-convergent is a special case of almost convergent topology.

(7.1) Definition: Let (X, L) and (Y,S) be topological spaces. For each pair of open sets $U \in L$ and $V \in S$, let

$$A(U,V) = \{f \in Y^X : f(U) \cap V \neq \emptyset\}$$

an almost convergent topology on Y^X is that topology which has as subbasis $\{A(U,V)\}$.

The following example provides a motivation to study the topology.

(7.2) Example: Let $f_n:[0,1] \to R$ (reals with the usual topology)

$$f = \begin{cases} 0 & \text{on } [0,1/2) \\ 1 & \text{on } [1/2,1] \end{cases}$$

$$f_n = \begin{cases} 0 & \text{on } [0,\ 2^n-1/2^{n+1}) \\ 2^n x - (2^n-1)/2 & \text{on } [(2^n-1)/(2^{n+1}),(2^n+1)/(2^{n+1})) \\ 1 & \text{on } [2^n+1/2^{n+1}\ ,\ 1] \end{cases}$$

for $n = 1,2,\ldots\ldots\ldots\ldots$

Since $f_n \notin P(1/2, S_r(1)) = \{f \in Y^X : f(1/2) \subset S_r(1)\}$, where $S_r(1)$ is the open sphere about 1 with the radius $r < 1/2$. $\{f_n\} \not\to f$ in the point open topology. However, $\{f_n\} \to f$ in the almost convergent topology (we denote as $\{f_n\} \xrightarrow{A} f$ from now on).

As the relation with other topologies we have

(7.3) Theorem: A-topology (Almost Convergent Topology) $\subset$ P-topology (pointwise convergent topology).

Proof: Let $A(U,V)$ be a subbasic open nbhd in A-topology and $f \in A(U,V)$ where $U \in L$, $V \in S$ and L, S are topologies in the domain and range spaces respectively. Then there exist $x \in U$ such that $f(x) \cap V = \emptyset$ which implies $f(x) \in V$ and $f \in P(x,V)$. Therefore, $P(x,V)$ is an open nbhd of f and contained in $A(U,V)$.

Combining the example (7.2) and the theorem (7.3) we have

(7.4) Corollary: Almost convergent topology is strictly smaller than the pointwise convergent topology except that they coincide when (X,L) is the discrete topology.

Proof: The first statement is an easy consequence of previous results. and if (X,L) is a discrete space then

$$P(x,V) = \{f \in Y^X : f(x) \in V\} = \{f \in Y^X : f(x) \cap V \neq \emptyset\}$$

and $x \in L$ which implies $P(x,V) = A(x,V)$.

As a separation axiom we have

(7.5) Theorem: The set of all continuous functions on X to Y which is denoted as $C(X)$ is T_1 with respect to the A-topology whenever (Y,S) is Hausdorff.

Proof: Let $f, g \in C(X)$ and $f \neq g$. Then there exist $x \in X$ such that $p = f(x) \neq g(x) = q$. Since Y is a Hausdorff space there exist $U,V \in S$ with $f(x) \in U$, $g(x) \in V$ and $U \cap V = \emptyset$. Since $f,g \in C(X)$ there exist $O_1, O_2 \in L$ such that $x \in O_1 \cap O_2$ and

$$f(O_1) \subset U \text{ and } g(O_2) \subset V$$

Then $(g(O_2) \cap U) \subset (V \cap U) = \emptyset$ and $(f(O_1) \cap V) \subset (V \cap U) = \emptyset$

Let $O = O_1 \cap O_2$ then $x \in O \in L$ and $g \notin A(O,U)$, $f \notin A(O,V)$ while $f \in A(O,U)$ and $g \in A(O,V)$.

It is well known that in the pointwise convergent topology $\lim f_n = f$ iff $\lim f_n(x) = f(x)$ for every $x \in X$. By Corollary (7.4) we expect a wider result in the A-topology. In fact we have the following example.

(7.6) Example:

$$f_1 = \begin{cases} 2x & \text{on } [0,1/2) \\ 2-2x & \text{on } [1/2,1] \end{cases}$$

$$f_2 = \begin{cases} 2^2x & \text{on } [0,1/2^2) \\ 2-2^2x & \text{on } [1/2^2, 1/2) \\ -2+2^2x & \text{on } [1/2,3/2^2) \\ 2^2-2^2x & \text{on } [3/2^2, 1] \end{cases}$$

$$f_n = \begin{cases} 2^nx & \text{on } [0,1/2^n) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^n-2^nx & \text{on } [2^n-1/2^n, 1] \end{cases}$$

Since dyadic fractions are dense in [0,1] $\{f_n\} \xrightarrow{A} f$ where

$$f = \begin{cases} 1 & \text{on rationals} \\ 0 & \text{on irrationals} \end{cases}$$

Moreover, let $\tilde{f} = 1$ and $\approx{f} = 0$, then $\{f_n\} \xrightarrow{A} \tilde{f}$ and $\{f_n\} \xrightarrow{A} \approx{f}$ where $\tilde{f}(X) \cap \approx{f}(X) = \emptyset$ and both $\tilde{f}$ and $\approx{f}$ are continuous functions. In fact if $\{f_n\} \xrightarrow{A} f$ and the graph of f is dense in the graph of $\approx{f}$ then $\{f_n\} \xrightarrow{A} \approx{f}$ also.

There is an interesting relation between A-topology and Skorokhod M-topology which he denoted as M_2-topology (see [23] p.266) in the space of all functions which are defined on the interval $[0,1]$ whose range space Y is a complete separable metric space, and which at every point have a limit on the left and are continuous on the right.

(7.7) Definition (Skorokhod): $R[(x_1, f(x_1)), (x_2, f(x_2))] = |x_1 - x_2| + d(f(x_1), f(x_2))$ where $d(f(x_1), f(x_2))$ is the distance of $f(x_1)$ and $f(x_2)$ in Y. $\{f_n\}$ is said to be M-convergent to f iff

$$\lim_{n \to \infty} \sup_{(x_1, f(x_1)) \in C(f)} \operatorname{Inf}_{(x_2, f_n(x_2)) \in C(f_n)} R[(x_1, f(x_1)), (x_2, f_n(x_2))] = 0$$

where $C(f) = \{(x, f(x)) : x \in [0,1]\}$.

Let $U_n = S(x, 1/n) = \{z: |x-z| < 1/n, z \in X\}$ and $V_n = S_d(f(x), 1/n) = \{y: d(f(x), y) < 1/n, y \in Y\}$ then $A = (U_n, V_n)$ is an element in the A-topology and $\{f\} \to f$ in the A-topology iff $G(f_\ell) \cap (U_n \times V_n) = \emptyset$ at each point $x \in X$ and $\ell \geq N_x$ for some fixed N_x i.e. $f \in A(U_n, V_n)$ for $\ell \geq N_x$. Therefore, Skorokhod M-topology is a special case of A-topology.

The almost convergent topology has an another aspect with respect to Fourier series. The following is well known in the theory of Fourier series.

Let $f(x)$ be a function defined in $-\pi \leq x \leq \pi$, and outside this interval defined by the eqation $f(x + 2\pi) = f(x)$. If $f(x)$ has a finite number of points of discontinuity of the first kind and a finite number of maxima and minima in the interval $-\pi \leq x \leq \pi$, then it can be replaced by the series

$$a_{0/2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

with

$$a_k = 1/\pi \int_{-\pi}^{\pi} f(t)\cos kt\,dt$$

$$b_k = 1/\pi \int_{-\pi}^{\pi} f(t)\sin kt\,dt \qquad (k = 0,1,2,\ldots)$$

which converge at every point $x = x_0$ of the interval to the value

$$\frac{f^+(x_0) + f^-(x_0)}{2}$$

Let f be as in the above statement. Then it is obvious that every open nbhd of $(x, f^+(x))$ and $(x, f^-(x))$ in $X \times Y$ (X, Y has the usual topology) has non-empty intersection with $C(\tilde{f})$ and $C(\approx\!f)$ where

$$\overset{\approx}{f}(x) = f^+(x) \quad , \quad \tilde{f} = f^-(x) .$$

The series

$$\{ S_n = a_{0/2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \}$$

converges to $\approx{f}$ and $\tilde{f}$ in the A-topology .

As a generalization of A-topology in the bitopological space we have the following.

(7.8) <u>Definition:</u> Let (X, L_1, L_2) and (Y, S_1, S_2) be bitopological spaces. <u>p-almost convergent topology on Y^X</u> is defined with the subbase

$$A_i(U,V) = \{f \in Y^X : f(U) \cap V \neq \emptyset\} .$$

where $U \in L_i$ and $V \in S_i$, $i = 1,2$. If α_i is generated by $\{A_i(U,V)\}$ then

$$(Y^X, \alpha_1, \alpha_2)$$

is said to be a <u>p-almost convergent bitopological function space.</u>

from (7.4) we have

(7.9) <u>Corollary:</u> $\alpha_i \subset P_i$, $i = 1,2$.

As a separation axiom in the space we discuss the following.

(7.10) <u>Theorem:</u> $(C_{S_{12}}^{L_{12}}, \alpha_1, \alpha_2)$ is p-T_1 whenever (Y, S_1, S_2) is p-T_2.

<u>Proof:</u> Let $f, g \in C_{S_{12}}^{L_{12}}$ and $f \neq g$ then there exists $x \in X$ such that

$p = f(x) \neq g(x) = q$. Since Y is p-Hausdorff there exist $U \in S_i$, $V \in S_j$, such that

$$f(x) \in U, \quad g(x) \in V \quad \text{and} \quad U \cap V = \emptyset .$$

where $i,j = 1,2$ and $i \neq j$. Since $f,g \in C_{S_{12}}^{L_{12}}$ there exist $O_1 \in L_1$, $O_2 \in L_2$ such that $x \in O_1 \cap O_2$ and $f(O_1) \subset U$ and $g(O_2) \subset V$.

Then

$$(f(O_1) \cap V) \subset U \cap V = \emptyset \quad \text{and}$$

$$(g(O_2) \cap U) \subset U \cap V = \emptyset$$

which implies

$$A_1(O_1,U) \in \alpha_1 \quad , \quad A_2(O,V) \in \alpha_2$$

$$f \in A_1(O_1,U) \quad \text{and} \quad g \notin A_1(O_1,U) \quad \text{while}$$

$$g \in A_2(O_2,V) \quad \text{and} \quad f \notin A_2(O,V) .$$

and the statement is proved.

Similarly we can show

(7.11) <u>Theorem:</u> $(C_{S_{12}}^{L_{12}}, \alpha_1,\alpha_2)$ is p-T_0 whenever (Y, S_1, S_2) is p-T_1.

NOTATIONS AND CONVENTIONS

Notations:

$A^{L_{12}}_{S_{12}}$	The set of all almost continuous functions $f:(X, L_1, L_2) \to (Y, S_1, S_2)$
$\bar{A}$	L_1(or S_1) - closure of A
$\bar{\bar{A}}$	L_2(or S_2)- closure of A
$\bar{A}^{L_i}$	L_i-closure of A
A^c	Complement of A
$A_i(U,V)$	$\{f \in Y^X : f(U) \cap V \neq \emptyset$, where $U \in L_i$, $V \in S_i\}$
α_i	Almost convergent topology which is generated by $A_i(U,V)$
$C^{L_{12}}_{S_{12}}$	The set of all p-continuous functions $f:(X, L_1, L_2) \to (Y, S_1, S_2)$
$\tilde{C}(K,U,V)$	$\{f \in Y^X : f(K) \subset U \cup V$, $f(K) \cap U \neq \emptyset \neq f(K) \cap V$, $U \in S_1$, $V \in S_2$ and K is (12)-connected set$\}$
C^*_{12}	The topology generated by $\{\tilde{C}(K,U,V)\}$
$C^{-2L_{12}}_{S_{12}}$	The set of all p-connected mappings $f:(X, L_1, L_2) \to (Y, S_1, S_2)$

$C_i(K,U,V)$	$\{f \in Y^X : f(K) \subset U \cup V,\ f(K) \cap U \neq \emptyset \neq f(K) \cap V,\ U, V \in S_i$ and K is p-connected set$\}$
$C(f)$	$\{(x,f(x)) : x \in X\}$
C_1	The first axiom of countability
C_2	The second axiom of countability
C topology	Connected open topology
ψ_i	The topology generated by $\{K(A,V) : A$ is a p-compact and $v \in S_i\}$
Γ_i	The topology generated by $\{C_i(K,U,V)\}$
G_U	$\{f \in Y^X : C(f) \subset U \in L \times S\}$
$\tilde{G}_S^{L_{12}}$	$\{Y^X, G_{L_1} \times S, G_{L_2} \times S\}$
$G_{L \times S}$	$\{G_U : U \in L \times S\}$
G topology	Graph topology
iff	If and only if
I	The set of all integers
κ_i(or K topology)	Compact open topology on $f : (X, L_i) \to (Y, S_i)$
$\tilde{K}(A,B)$	$\{f \in Y^X : f(A) \subset B\}$

LSC_L	Lower semi continuous with respect to L
LUX	The set of all real valued functions that are both USC_{L_1} and LSC_{L_2}
N	The set of all positive integers
p-	Pairwise
PC	Point open topology
p-K	p-compact open topology
$p\text{-}K(C_{S_{12}}^{L_{12}})$	$C_{S_{12}}^{L_{12}}$ with p-K topology
p-K(ULX)	ULX with p-K topology
p-q	Pseudo quasi-metric
P_i	PC topology with range space (Y,S_i)
R	The set of all real numbers
$L_i(V)$	Adjoint topology of L_i w.r.t. V, i.e. $\{U \cup V : U \in L_i\}$
ULX	The set of all real valued functions that are both USC_{L_1} and LSC_{L_2}
USC_L	Upper semi continuous w.r.t. L
w.r.t.	With respect to

$\vee(\wedge)$ Sup (Inf)

Conventions:

We will often use the following:

(1) $L_i(S_i)$ will mean a topology for $X(Y)$

(2) If it is clear from the context we frequently do not use any subscript: e.g. ψ instead of ψ_i

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